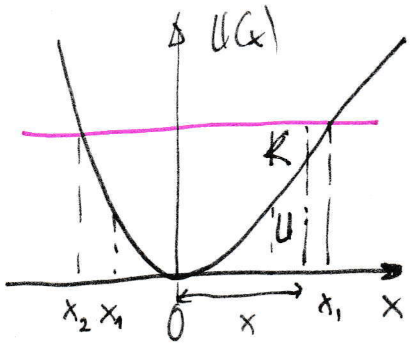
Drgania – ruch harmoniczny $\nu = \frac{1}{T}$

- wykorzystuje sprężyste właściwości materiałów lub ośrodka
- ruch okresowy → powtarzający się w j. czasu (periodyczny)
- ruch drgający (wibracyjny, oscylacyjny) → powtarza się po tej samej drodze

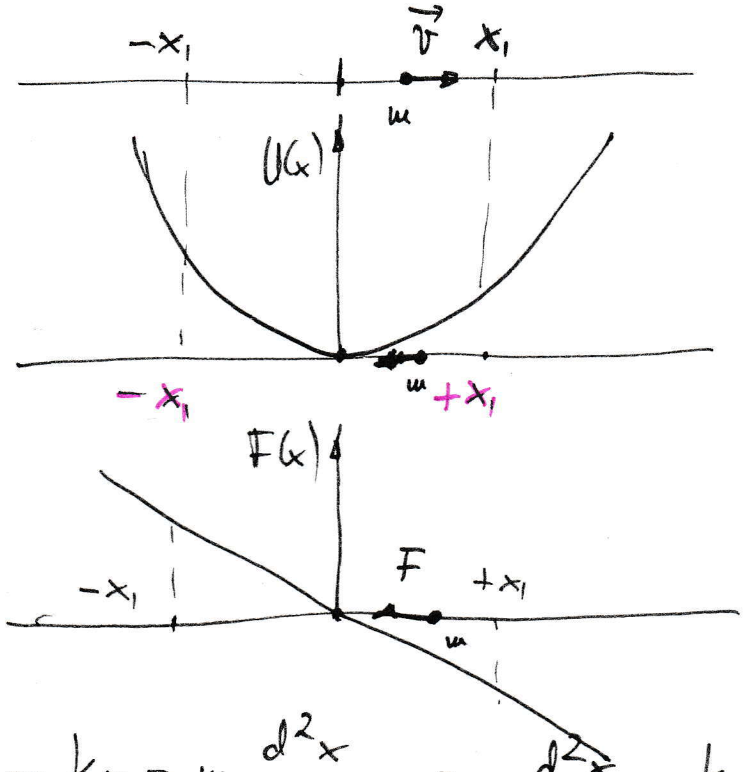


$$\boxed{E = U + K}$$

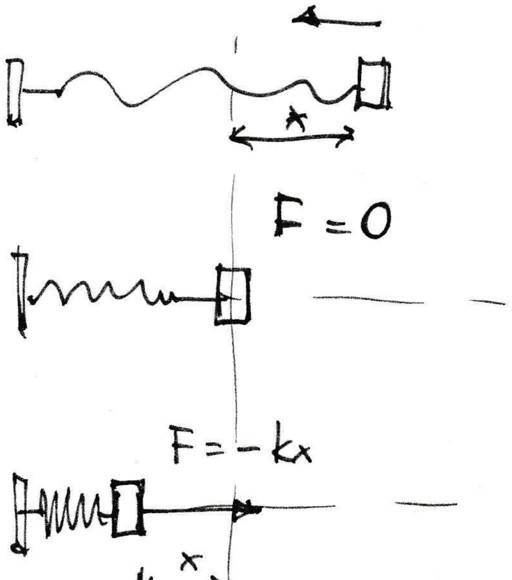
$$U(x) = \frac{1}{2} k x^2$$

$$F = -\frac{dU}{dx}$$

$$F(x) = -\frac{dU}{dx} = -kx$$



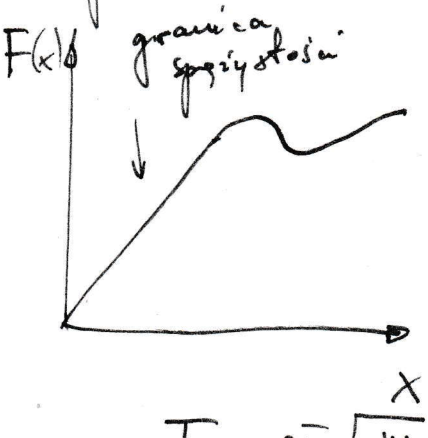
Sprężyna



$$-kx = m\frac{d^2x}{dt^2} \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

r. oscylatora harmonicznego

prawo Hooke'a



$$x(t) = A\cos(\omega t + \varphi)$$

$$\frac{dx}{dt} = -\omega A \sin(\omega t + \varphi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \varphi)$$

$$\Rightarrow -\omega^2 A\cos(\omega t+\varphi) = -\frac{k}{m}A\cos(\omega t+\varphi)$$

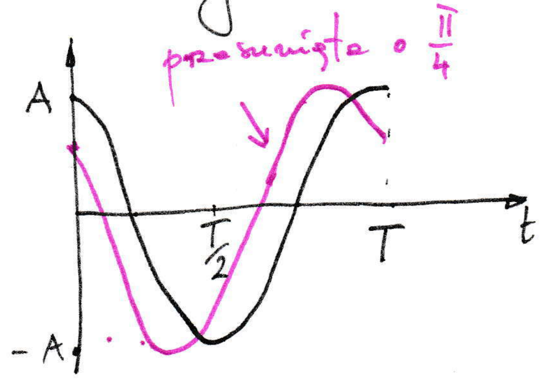
$$\Downarrow$$

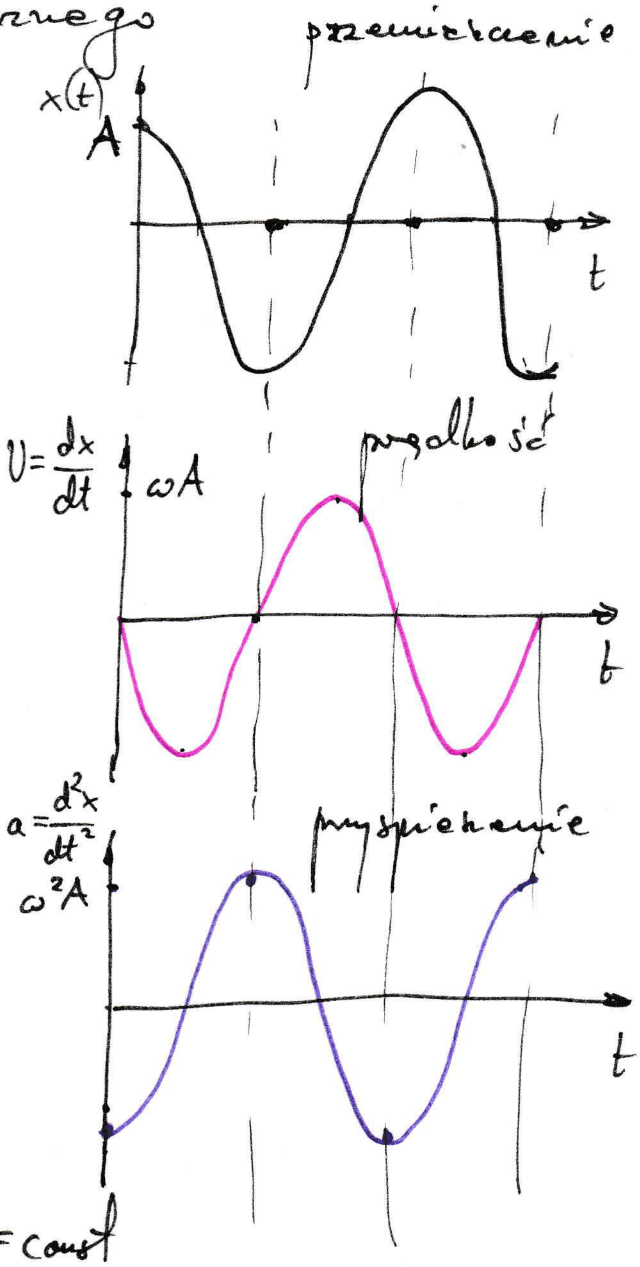
$$\omega^2 = \frac{k}{m}$$

$T = 2\pi\sqrt{\frac{m}{k}}$ okres

częstość drgań własnych

2 Cechy ruchu harmonicznego





$\boxed{\text{Energia}}$ $\qquad \omega^2 = \frac{k}{m}$

$$x(t) = A\cos(\omega t + \varphi)$$

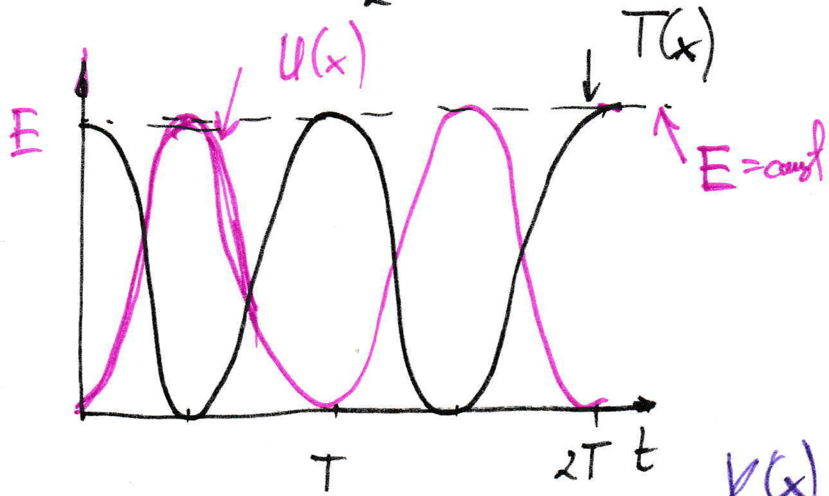
$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \varphi)$$

$$v(t) = -\omega A\sin(\omega t + \varphi)$$

$$T(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\sin^2(\omega t + \varphi)$$

$$= \frac{1}{2}kA^2\sin^2(\omega t + \varphi)$$

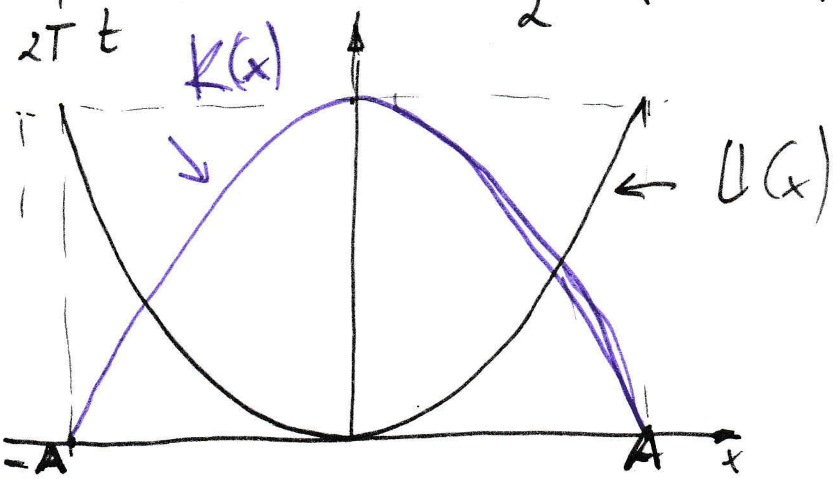
$$E = U + T = \frac{1}{2}kA^2(\underbrace{\sin^2\ldots + \cos^2\ldots}_{=1}) = \text{const}$$



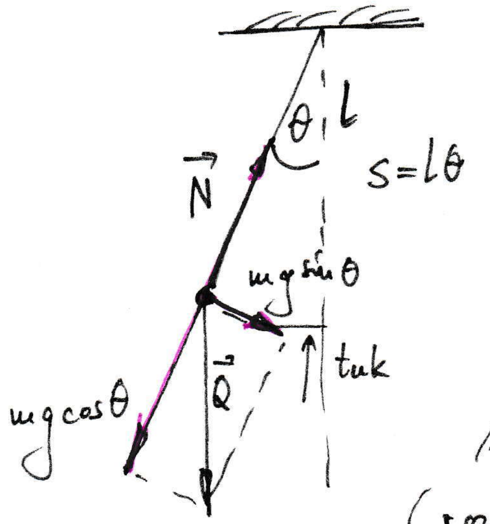
$$U(x) = \frac{1}{2}kx^2$$

$$T(x) = \frac{1}{2}kA^2\sin^2(\omega t + \varphi) =$$

$$= \frac{1}{2}k(A^2 - x^2)$$



3 Wahadło matematyczne



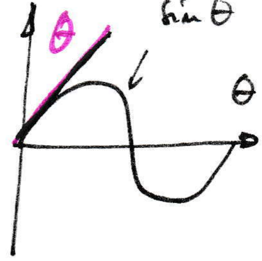
$s = l\theta$

$$F = -mg\sin\theta$$

$$F = m\frac{d^2s}{dt^2} = ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}$$ trudne do rozwiązania analit.

ALE! małe drgania $\sin\theta \approx \theta$ (rozwijamy w szereg wokół $\theta = 0$)



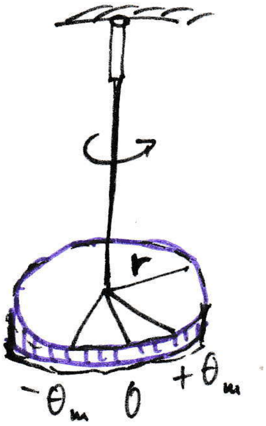
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$ równanie oscylatora

$$\boxed{\omega^2 = \frac{g}{l}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

poprawki dla dużych kątów: $T = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{2^2}\sin^2\frac{\theta_m}{2} + \frac{1}{2^2}\frac{3^2}{4^2}\sin^4\frac{\theta_m}{2} + \ldots\right)$

dla $\theta_m = 15°$ błąd ok. 0.5%

wahadło torsyjne (np. doświadczenie Cavendisha)



moment siły τ ~ kąta skręcenia θ (p. Hooke'a)

$$\boxed{\tau = -\kappa\theta}$$

stała skręcenia zależy od materiału

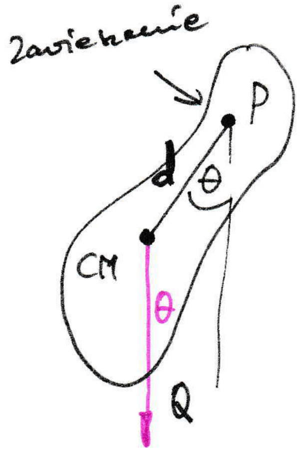
$$\tau = I\varepsilon = I\frac{d^2\theta}{dt^2}$$

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta \Rightarrow \boxed{\frac{d^2\theta}{dt^2} + \frac{\kappa}{I}\theta = 0}$$ r. oscylatora

$$\omega^2 = \frac{\kappa}{I} \quad \text{lub} \quad T = 2\pi\sqrt{\frac{I}{\kappa}}$$

I – moment bezwładności krążka + pręta (można pominąć) $\quad I_k = \frac{1}{2}mr^2$

#4 wahadło fizyczne



podobnie jak dla w. matem. rozważamy małe drgania $\sin\theta \approx \theta$

$$\tau = -Mgd\sin\theta \approx -Mgd\theta \quad \text{lub} \quad \tau = -\varkappa\theta$$

$$\varkappa = Mgd$$

moment siły

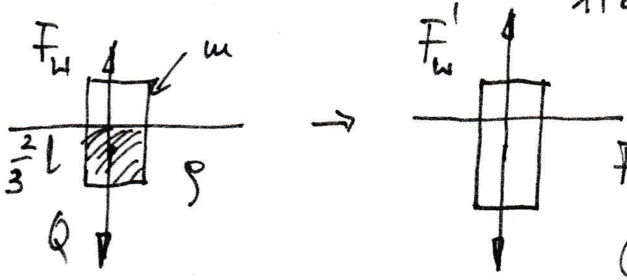
$$\tau = I\frac{d^2\theta}{dt^2} = -Mgd\,\theta$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{Mgd}{I}\theta = 0} \qquad \boxed{\omega^2 = \frac{Mgd}{I}} \qquad T = 2\pi\sqrt{\frac{I}{Mgd}}$$

- szczególny przypadek: wahadło mat.

$$\omega = \sqrt{\frac{Mgd}{I}} \;:\; I = Md^2 \;:\; \Rightarrow \omega^2 = \sqrt{\frac{Mgd}{Md^2}} = \sqrt{\frac{g}{d}} \;!$$

•• klocek w wodzie:



$F_w' > Q$

$F_w' = \left(\frac{2}{3}l + x\right)A\varrho g$

równowaga

$$F_w = Q \rightarrow \frac{2}{3}lA\varrho g = mg$$

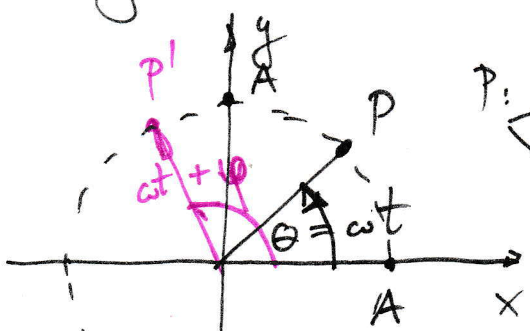
$Q = mg$

$$\Delta F = F_w' - Q = xA\varrho g = -m\frac{d^2x}{dt^2}$$

$m = AL\varrho_k$

$$\boxed{\frac{d^2x}{dt} + \frac{A\varrho g}{m}x = 0} \Rightarrow \omega^2 = \frac{A\varrho g}{m} = \frac{\cancel{A}\varrho g}{\cancel{A}L\varrho_k} = \frac{g}{L}\frac{\varrho}{\varrho_k}$$

drgania a ruch po okręgu



we współrz. biegunowych

$$P: x(t) = A\cos\omega t \quad ; \quad P': x(t) = A\cos(\omega t + \varphi)$$

lub

$$y(t) = A\sin\omega t \qquad y(t) = A\sin(\omega t + \varphi)$$

$$r = \sqrt{x^2 + y^2} = A$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(-\omega A\sin\omega t)^2 + (\omega A\cos\omega t)^2} = \omega A$$

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(-\omega^2 A\cos\omega t)^2 + (-\omega^2 A\sin\omega t)^2} = \omega^2 A$$

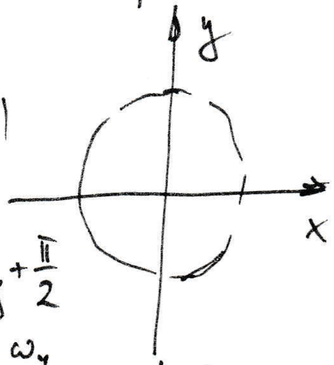
5 Składanie ruchów harmonicznych (lusterka!)

drgania w kierunkach prostopadłych

$$x(t) = A_x \cos(\omega_x t + \varphi_x) \qquad y(t) = A_y \cos(\omega_y t + \varphi_y)$$

np. $\Delta\varphi = 0$ oraz $\omega_x = \omega_y \Rightarrow y = \frac{A_y}{A_x} x$ (d. liniowe)

$\frac{A_y}{A_x} = 1$



$\varphi_x = \varphi_y + \frac{\pi}{2}$

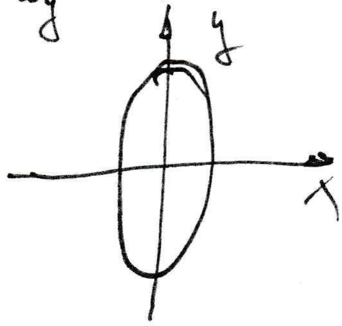
$\omega_x = \omega_y$

$$x(t) = A_0 \sin(\omega t)$$
$$y(t) = A_0 \cos(\omega t)$$
$$\Rightarrow \frac{x^2}{A^2} + \frac{y^2}{A^2} = \sin^2\omega t + \cos^2\omega t = 1$$
$$x^2 + y^2 = A^2 \quad \text{(koło)}$$

figury Lissajous

to samo ale $A_x \neq A_y$

czyli $\left(\frac{x}{A_x}\right)^2 + \left(\frac{y}{A_y}\right)^2 = 1$ (elipsa)



Ruch harmoniczny tłumiony

$F = ma$ siła tłumiąca ~ prędkości

$$F_{tł} \sim -bv = -b\frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$\boxed{\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \overbrace{\frac{k}{m}}^{\omega^2} x = 0}$$

$$\lambda^2 + \frac{b}{m}\lambda + \omega^2 = 0$$
$$\Delta = \left(\frac{b}{m}\right)^2 - 4\omega^2$$
$$\lambda_{1,2} = \frac{-\frac{b}{m} \pm i2\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}}{2} = -\frac{b}{2m} \pm i\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$$

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$$x(t) = A e^{-\frac{bt}{2m}} \cos(\omega' t + \varphi)$$

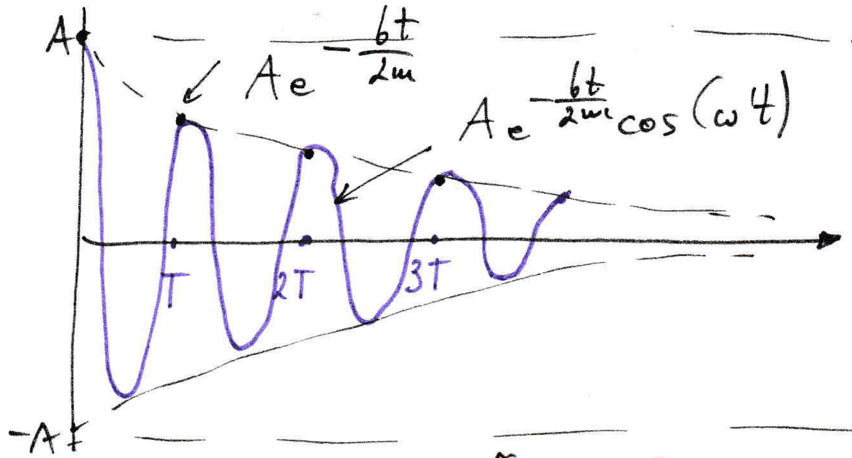
$$\boxed{\omega' = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}}$$

bez tłumienia

$\omega' = \omega$

rozwiązanie równania różniczkowego zwyczajnego ze stałymi współczynnikami!

6



$\frac{b}{2m}$ – współczynnik tłumienia

$$x(t) = Ae^{-\frac{bt}{2m}}\cos(\omega' t)$$

$$x(t+T) = Ae^{-\frac{b(t+T)}{2m}}\cos[\omega'(t+T)]$$

$$\cos(\omega' t) = \cos[\omega'(t+T)]$$

$$\frac{x(t)}{x(t+T)} = \frac{\cancel{Ae^{-\frac{bt}{2m}}\cos(\omega' t)}}{\cancel{Ae^{-\frac{b}{2m}t}}\,e^{-\frac{bT}{2m}}\,\cancel{\cos[\omega'(t+T)]}} = e^{\frac{bT}{2m}}$$

$$\frac{bT}{2m} = \ln\frac{x(t)}{x(t+T)} \Rightarrow \frac{b}{2m} = \frac{1}{T}\ln\frac{x(t)}{x(t+T)}$$

logarytmiczny dekrement tłumienia

Drgania wymuszone i rezonans

$$\omega = \sqrt{\frac{k}{m}} \quad \text{lub} \quad \omega' = \sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{k}{m}x = \boxed{F_m \cos\omega'' t}$$

siła zewnętrzna

$$x(t) = \frac{F_m}{\sqrt{m^2(\omega''^2 - \omega^2)^2 + b^2\omega''^2}}\sin(\omega'' t - \varphi)$$

$$\boxed{\omega_r = \omega_0}$$

lub

$$\boxed{\omega_r = \omega'}$$

Rezonans